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The Omega Limit Sets of Ray-Contractive Operators

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In this paper, we study the omega limit sets of a ray-contractive map T without the compactness assumption on the orbits. If there exists a compact subset Ω of $\overset{\circ}{K}$ (the interior of a closed convex cone K) such that $T(\Omega) = \Omega$ and $T(K - \{0\}) \subset \overset{\circ}{K}$, then we are able to prove that $\omega(x)$ either consists of a single point or forms a cycle 2 periodic orbit for any $x \in \overset{\circ}{K} - \Omega$. © 2001 Academic Press

1. INTRODUCTION

Let B be a Banach space which is partially ordered by a closed convex cone K. We say $x \leq y$ if $y - x \in K$, $x < y$ if $y - x \in K \backslash \{0\}$, and $x \ll y$ if $\overset{\circ}{K} \neq \emptyset$ and $y - x \in \overset{\circ}{K}$. Throughout this paper, we will assume that $\overset{\circ}{K} \neq \emptyset$ and the norm is monotone; i.e., $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. Let $x, y \in K \backslash \{0\}$. x and y are comparable if there exist positive numbers λ and μ such that $\lambda x \leq y \leq \mu x$. This equivalence relationship splits $K \backslash \{0\}$ into disjoint parts of K and $\overset{\circ}{K}$ is a part if $\overset{\circ}{K} \neq \emptyset$.

On a part C of K, Thompson [13] defined a metric

$$d(x, y) = \ln\{\max[M(x/y),\ M(y/x)]\},$$

where $x, y \in C$, and

$$M(x/y) = \inf\{\lambda \in R : x \leq \lambda y\}.$$

This metric is called the Thompson metric or the part metric. It is a very useful tool for studying the mappings that satisfy certain homogeneous-type conditions, because they are essentially contraction-type mappings in (C, d),

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and various contraction-type fixed point theorems can be applied (see [1–3, 6–11, 13] and the references therein). However, for a mapping $T: C \to C$ which is only nonexpansive in the part metric, i.e., $d(Tx, Ty) \leq d(x, y)$, there may not be a unique fixed point and additional conditions may be needed to study the dynamics of T.

Recently, Takáč [12] introduced the notion of a ray-contractive mapping. A mapping $T: K \to K$ is said to be ray-contractive if it satisfies

(i) $d(Tx, Ty) \leq d(x, y)$; and

(ii) if $d(T^n x, T^n y) = d(x, y)$ for all $n = 1, 2, 3, \ldots$, then $y = \lambda x$ for some $\lambda \in (0, \infty)$.

Nussbaum [10, Proposition 3.1, p. 100] also considered a mapping that satisfied some similar properties.

For convenience, we quote a special case of Takáč's main result as follows.

THEOREM 1.1 [12, Theorem 3.1]. *Let C be a part of K and let Ω be a nonempty compact subset in (C, d). Suppose $T(\Omega) = \Omega$ and T is ray-contractive under d on Ω. Then there exist $\gamma \in [1, \infty)$ and $e \in C$ such that $\gamma^{-1}e, \gamma e \in \Omega$,*

$$\Omega \subset [\gamma^{-1}, \gamma]e = \{\alpha e \in C : \alpha \in [\gamma^{-1}, \gamma]\},$$

and for either $\epsilon = 1$ or $\epsilon = -1$, we have $T(\alpha e) = \alpha^{\epsilon} e$ whenever $\alpha \in [\gamma^{-1}, \gamma]$ and $\alpha e \in \Omega$. In particular, $T^2 x = x$ for all $x \in \Omega$.

Using Theorem 1.1 and under the additional compactness assumptions, Takáč discussed the asymptotic behavior of T at any $x \in K \backslash \{0\}$.

Let $\omega(x)$ be the omega limit set of $x \in K$ with respect to T, i.e., $y \in \omega(x)$ if and only if there is a sequence of positive integers $n_i \to \infty$ such that $T^{n_i} x \to y$. The purpose of this paper is to study the properties of $\omega(x)$ for any $x \in \overset{\circ}{K} \backslash \{0\}$. It turns out that the structure of $\omega(x)$ is quite simple even without the assumption that $\omega(x)$ is contained in the compact subset Ω of (C, d).

2. MAIN RESULTS

We will need the following lemma which was proved by Krause and Nussbaum [9].

LEMMA 2.1 [9, Lemma 2.3]. (i) *Let $x,\ y \in \overset{\circ}{K}$ and let $r > 0$ be a number such that the closed balls of radius r and centers x and y are contained in K. Then*

$$d(x,\ y) \leq \ln\left(1 + \frac{\|x - y\|}{r}\right). \tag{1}$$

(ii) *If the norm is monotone on K, then for $x,\ y \in K - \{0\}$,*

$$\|x - y\| \leq (2e^c - e^{-c} - 1)\min\{\|x\|, \|y\|\} \tag{2}$$

for any $c \geq d(x, y)$.

(2) is slightly stronger than [9, (2.5)], and it is true because $f(t) = 2e^t - e^{-t} - 1$ is an increasing function. This observation is useful in our paper.

For $x \in K$ and for $S \subset K$, let

$$\operatorname{dist}(x, S) = \inf\{d(x, s) : s \in S\}.$$

We first establish some results that concern the structure of $\omega(x)$ of a nonexpansive mapping.

THEOREM 2.2. *Let T: $K \to K$ with $T(K\backslash\{0\}) \subset \overset{\circ}{K}$ be nonexpansive in $(\overset{\circ}{K}, d)$. Suppose there is a nonempty compact subset Ω of $(\overset{\circ}{K}, d)$ such that $T(\Omega) \subset \Omega$. Then for all $x \in K$, $\lim_{n\to\infty} \operatorname{dist}(T^n x, \Omega) = c$ exists and $c \in [0, \operatorname{dist}(x, \Omega)]$. Moreover, for each $x' \in \omega(x)$, $\operatorname{dist}(x', \Omega) = c$ and there is a $w \in \Omega$ such that $d(T^k x', T^k w) = c$ for $k = 0, 1, 2, \ldots$.*

Proof. For all $n \geq 0$ there exists $w_n \in \Omega$ such that $d(T^n x, w_n) = \operatorname{dist}(T^n x, \Omega)$. Since

$$d(T^n x, w_n) \leq d(T^n x, Tw_{n-1}) \leq d(T^{n-1} x, w_{n-1}),$$

$\{d(T^n x, w_n)\}$ is nonincreasing and there exists $0 \leq c \leq d(x, w_0)$ such that $\lim_{n\to\infty} d(T^n x, w_n) = c$.

Let $x' \in \omega(x)$. We claim that $x' \neq 0$. To prove this, put $c_0 = d(x, w_0)$. By (2),

$$\|T^n x - w_n\| \leq (2e^{c_0} - e^{-c_0} - 1)\|T^n x\|. \tag{3}$$

Assume that $x' = 0$. Then there exists n_i such that $\lim_{i\to\infty} \|T^{n_i} x\| = 0$. Since Ω is compact, without loss of generality, we can assume that there exists $w \in \Omega$ with $\lim_{i\to\infty} \|w_{n_i} - w\| = 0$. Using (3), we have $\|w\| \leq 0$, which is impossible. The claim is proved.

Now let T^{n_i} be a subsequence such that $\lim_{i\to\infty} \|T^{n_i} x - x'\| = 0$. Since T is an isometry of $\omega(x)$ onto $\omega(x)$ [4, Theorem 1], $T(K\backslash\{0\}) \subset \overset{\circ}{K}$ implies that $x' \in \overset{\circ}{K}$. Thus $\lim_{i\to\infty} d(T^{n_i} x, x') = 0$ by (1). On the other hand, we have $\lim_{i\to\infty} d(T^{n_i} x, w_{n_i}) = c$. Without loss of generality, we can assume $\lim_{i\to\infty} d(w_{n_i}, w) = 0$ for some $w \in \Omega$, because Ω is compact in $(\overset{\circ}{K}, d)$. Thus $d(x', w) = c$. Furthermore, for any integer $k > 0$,

$$d(T^{n_i+k} x, w_{n_i+k}) \leq d(T^{n_i+k} x, T^k w_{n_i}).$$

Letting $i \to \infty$,

$$c \leq d(T^k x', T^k w)$$

by the continuity of T^k and d. Since

$$d(T^k x', T^k w) \leq d(x', w) = c,$$

we have

$$d(T^k x', T^k w) = c, \qquad k = 0, 1, 2, \ldots.$$

Finally, $d(x', w) = \operatorname{dist}(x', \Omega)$, because $d(T^{n_i} x, w_{n_i}) = \operatorname{dist}(T^{n_i} x, \Omega)$, and letting $i \to \infty$ completes the proof. ∎

If Ω in Theorem 2.2 is a single point, then we get the following corollary which is related to the results obtained by Kloeden and Rubinov [5, Theorem 4.1].

COROLLARY 2.3. *Let* $T\colon K \to K$ *with* $T(K \backslash \{0\}) \subset \mathring{K}$ *be nonexpansive in* $(\mathring{K}, d)$. *Suppose there is a* $u \in (\mathring{K}, d)$ *such that* $Tu = u$. *Then for each* $x \in \mathring{K}$ *there is a* $c \in [0, d(x, u)]$ *such that*

$$\omega(x) \subset \{y : d(T^k y, T^k u) = c \quad \textit{for } k = 0, 1, 2, \ldots\}.$$

EXAMPLE. Let $K = R^r_+ = \{x = (x_1, \ldots, x_r) : x_i \geq 0\}$. Then for any $x, y \in \mathring{K}$,

$$d(x, y) = \ln \max\left\{\max_{1 \leq i \leq r} \frac{x_i}{y_i}, \max_{1 \leq i \leq r} \frac{y_i}{x_i}\right\}.$$

Let $u = (u_1, \ldots, u_r) \in \mathring{K}$ be a fixed point of a nonexpansive mapping T on $(\mathring{K}, d)$. Suppose $T(K \backslash \{0\}) \subset \mathring{K}$. Note that $\{y : d(y, u) = c\}$ consists of the surfaces of the hypercube $\{y : e^{-c} u \leq y \leq e^c u\}$. Corollary 2.3 tells us that for any $x \in \mathring{K}$, $\omega(x)$ and all its iterates $T^n \omega(x)$ are contained in the surface of the hypercube $y : l^{-1} u \leq y \leq lu$, where

$$l = \max\left\{\max_{1 \leq i \leq r} \frac{x_i}{u_i}, \max_{1 \leq i \leq r} \frac{u_i}{x_i}\right\}.$$

The following theorem gives a complete description of $\omega(x)$ for any $x \in \mathring{K}$ under a ray-contractive mapping T.

THEOREM 2.4. *Let T: $K \to K$ with $T(K\backslash\{0\}) \subset \overset{\circ}{K}$ be ray-contractive. Suppose there is a nonempty compact subset Ω of $(\overset{\circ}{K}, d)$ such that $T(\Omega) = \Omega$. Then there exists a $u \in \overset{\circ}{K}$ such that for $x \in \overset{\circ}{K}$,*

$$\omega(x) \subset \{\alpha u : \alpha > 0\}. \tag{4}$$

Moreover for $x \in \overset{\circ}{K} \backslash\Omega$, $\lim_{n\to\infty} \operatorname{dist}(T^n x, \Omega) = c$ exists and for $x' \in \omega(x)$, there is a $w \in \Omega$ such that $d(x', w) = c$ and $x' = e^c w$ or $x' = e^{-c} w$. Furthermore, $\omega(x)$ either is a single point, which is a fixed point of T, or forms a 2-cycle of T.

Proof. Let $x \in \overset{\circ}{K}$. Then by Theorem 2.2, $\lim_{n\to\infty} \operatorname{dist}(T^n x, \Omega) = c$ exists. Let $x' \in \omega(x)$. Then also by Theorem 2.2, $c = \operatorname{dist}(x', \Omega)$ and there exists $w \in \Omega$ such that

$$d(T^k x', T^k w) = c \quad \text{for } k = 0, 1, 2, \ldots.$$

Because T is ray-contractive, there exists $\lambda_k > 0$ such that

$$T^k x' = \lambda_k T^k w, \qquad k = 0, 1, 2, \ldots. \tag{5}$$

Now $d(\lambda_k T^k w, T^k w) = c$ implies that $c = \ln\{\max\{1/\lambda_k, \lambda_k\}\}$. Hence $T^k x' = e^c T^k w$ or $T^k x' = e^{-c} T^k w$. Using Theorem 1.1 [12, Theorem 3.1], there exist $u \in \overset{\circ}{K}$ and $\alpha > 0$ such that $w = \alpha u$ and $T(\alpha u) = \alpha^\epsilon u$, where $\epsilon = 1$ or $\epsilon = -1$. In particular, $T^2(\alpha u) = \alpha u$. Thus $x' = \lambda_0 \alpha u$ and (4) is proved.

Noting that w is either a fixed point or a cycle 2 periodic point of T (Theorem 1.1) and that $\lambda_k = e^c$ or e^{-c} implies that $\lambda_k = \lambda_0$ or λ_0^{-1}, (5) implies that $\{T^k x', k = 0, 1, 2, \ldots\}$ consists of at most four different points. Hence $\omega(x')$ is a finite set and consequently compact. By Theorem 1 of [4], $\omega(x)$ is also compact due to $\omega(x) = \omega(x')$. Again using Theorem 1.1, any point in $\omega(x)$ is either a fixed point or a cycle 2 periodic point of T. We discuss it in two cases.

Case 1. If $\omega(x)$ contains a fixed point, then $\omega(x) = \{x'\}$. To see this, assume that $T^k x'$ is a fixed point of T with $k > 0$. Since T is an isometry of $\omega(x)$ onto $\omega(x)$ [4, Theorem 1], $d(T^{k-1}x', T^k x') = d(T^k x', T^{k+1}x') = 0$; i.e., $T^{k-1}x'$ is also a fixed point of T. By induction, x' is a fixed point. Hence $\omega(x) = \omega(x') = \{x'\}$.

Case 2. If $\omega(x)$ contains a cycle 2 periodic point, then $\omega(x) = \{x', Tx'\}$. To see this, assume that $T^k x'$ is a cycle 2 periodic point of T with $k > 0$. Whereas $d(T^{k-1}x', T^{k+1}x') = d(T^k x', T^{k+2}x') = 0$, $T^{k-1}x'$ is also a cycle 2 periodic point of T. Again by induction, x' is a cycle 2 periodic point. Hence $\omega(x) = \omega(x') = \{x', Tx'\}$. ∎

3. SOME EXAMPLES

In this section, we will discuss some mappings which are ray-contractive. These mappings include some known mappings which appear in the literature.

Let C be a part of K. We say $T: C \to C$ is subhomogeneous if $T(\alpha x) \geq \alpha Tx$ for all $(\alpha, x) \in (0,1) \times C$. It is well known that a monotone subhomogeneous mapping is nonexpansive under (C, d) (e.g., see [12, Proposition 2.4]).

PROPOSITION 3.1. *Let C be a part of K and $T: C \to C$ is monotone subhomogeneous. If there exist a function $\varphi: (0,1) \to (0,1)$ with $\varphi(t) > t$ and a positive integer $n(x)$, which depends on $x \in C$, such that*

$$T^{n(x)}(tx) \geq \varphi(t) T^{n(x)} x, \qquad (t, x) \in (0,1) \times C, \tag{6}$$

then T is ray-contractive on (C, d).

Proof. Assume that $d(x, y) = \ln \alpha > 0$, but we have neither $y = \alpha x$ nor $y = \alpha^{-1} x$. Then $\alpha^{-1} x < y < \alpha x$. Using (6), there exist positive integers $n(x)$ and $n(y)$ such that

$$T^{n(x)} y \geq T^{n(x)}(\alpha^{-1} x) \geq \varphi(\alpha^{-1}) T^{n(x)} x$$

and

$$T^{n(y)} x \geq T^{n(y)}(\alpha^{-1} y) \geq \varphi(\alpha^{-1}) T^{n(y)} y.$$

Since T is monotone subhomogeneous, we have

$$T^{n(x)n(y)} y \geq T^{n(y)}(\varphi(\alpha^{-1}) T^{n(x)} x) \geq \varphi(\alpha^{-1}) T^{n(x)n(y)} x$$

and

$$T^{n(x)n(y)} x \geq T^{n(x)}(\varphi(\alpha^{-1}) T^{n(y)} y) \geq \varphi(\alpha^{-1}) T^{n(x)n(y)} y.$$

Hence

$$d(T^{n(x)n(y)} x, T^{n(x)n(y)} y) \leq \ln(\varphi(\alpha^{-1})^{-1}) < \ln \alpha = d(x, y).$$

It follows that T must be ray-contractive. ∎

It is clear that ascending mapping, introduced by Krause [6, 7], and the mappings studied in [3] satisfy the condition (6).

We say a mapping $T: C \to C$ is strongly primitive (cf. [8, p. 184]) if there exists $s(x) \geq 1$, which depends on x, such that

$$x < y \quad \text{implies} \quad T^{s(x)} x \ll T^{s(x)} y, \tag{7}$$

where $x, \ y \in C$.

PROPOSITION 3.2. *Let C be a part of K and T: $C \to C$ is monotone subhomogeneous. Then T is ray-contractive if one of the following conditions is satisfied.*

(i) *T is strongly primitive.*

(ii) *For any $(t, x) \in (0,1) \times C$, there exists a positive integer $n(x)$, which depends on x, such that $T^{n(x)}(tx) \gg tT^{n(x)}x$.*

Proof. Let us suppose that (i) is satisfied. Assume that $d(x, y) = \ln \alpha > 0$, but we have neither $y = \alpha x$ nor $y = \alpha^{-1}x$. Then $\alpha^{-1} < y < \alpha x$. By (7), there exist positive integers $s(\alpha^{-1}x)$ and $s(\alpha^{-1}y)$ such that

$$T^{s(\alpha^{-1}x)}y \gg T^{s(\alpha^{-1}x)}(\alpha^{-1}x) \geq \alpha^{-1}T^{s(\alpha^{-1}x)}x$$

and

$$T^{s(\alpha^{-1}y)}x \gg T^{s(\alpha^{-1}y)}(\alpha^{-1}y) \geq \alpha^{-1}T^{s(\alpha^{-1}y)}y.$$

Hence there exists $\beta \in (\alpha^{-1}, 1)$ such that

$$T^{s(\alpha^{-1}x)}y \geq \beta T^{s(\alpha^{-1}x)}x$$

and

$$T^{s(\alpha^{-1}y)}x \geq \beta T^{s(\alpha^{-1}y)}y.$$

It follows that

$$d(T^{s(\alpha^{-1}x)s(\alpha^{-1}y)}x, T^{s(\alpha^{-1}x)s(\alpha^{-1}y)}y) \leq \ln \beta^{-1} < \ln \alpha = d(x, y),$$

and we conclude that T is ray-contractive.

In a similar spirit, we can prove that T is also ray-contractive when (ii) is satisfied. ∎

Proposition 3.2 extends [12, Proposition 2.4].

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